Proof: $\frac{d}{dt} H(t) = \int d\vec{p}\, d\vec{q}\; \partial_t [f \ln f] = \int d\vec{p}\, d\vec{q}\, (\ln f + 1)\, \partial_t f$ (1)

$$= \int \underbrace{d\vec{p}\, d\vec{q}}_{= d\Gamma} \ln f \cdot \partial_t f + \partial_t \underbrace{\underbrace{\int d\vec{p}\, d\vec{q}\, f}_{N}}_{=0}$$

Let us now use the BE to replace $\partial_t f$:

$$\frac{d}{dt} H(t) = \underbrace{\int d\Gamma \ln f \left[\frac{\partial H}{\partial \vec{q}} \cdot \underbrace{\frac{\partial f}{\partial \vec{p}}}_{\text{IBP}} - \frac{\partial H}{\partial \vec{p}} \cdot \underbrace{\frac{\partial f}{\partial \vec{q}}}_{\text{IBP}} \right]}_{1} + \underbrace{\int d\vec{p}_1\, d\vec{q}\, d\vec{p}_2\, d\sigma\, |\vec{v}_1 - \vec{v}_2| \left(f_1' f_2' - f_1 f_2 \right) \ln f_1}_{2}$$

$$(1) = \int d\Gamma \left[-\frac{1}{\cancel{f}} \frac{\partial f}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{q}} \cancel{f} + \frac{1}{\cancel{f}} \frac{\partial f}{\partial \vec{q}} \cdot \frac{\partial H}{\partial \vec{p}} \cancel{f} \right] = \int d\Gamma\, - \frac{\partial}{\partial \vec{p}} \cdot \left(f \frac{\partial H}{\partial \vec{q}} \right) + \frac{\partial}{\partial \vec{q}} \cdot \left(f \frac{\partial H}{\partial \vec{p}} \right)$$

$$= 0$$

In (2), $\vec{p}_1$ & $\vec{p}_2$ are dummy variables so

((2) with $\vec{p}_1, \vec{p}_2$) $= \frac{1}{2}$ ((2) with $\vec{p}_1, \vec{p}_2$) $+ \frac{1}{2}$ ((2) with $\vec{p}_2, \vec{p}_1$)

$$(2)_a = \frac{1}{2} \int d\vec{p}_1\, d\vec{q}\, d\vec{p}_2\, \underbrace{d\sigma(\vec{p}_1, \vec{p}_2 \to \vec{p}_1{}', \vec{p}_2{}')}\; \underbrace{|\vec{v}_1 - \vec{v}_2|} \left(f_1' f_2' - f_1 f_2 \right) \left[\ln f_1 + \ln f_2 \right]$$

$$= d\sigma(\vec{p}_1{}', \vec{p}_2{}' \to \vec{p}_1, \vec{p}_2)\; |\vec{v}_1{}' - \vec{v}_2{}'| \left(f_1' f_2' - f_1 f_2 \right) \left[\ln f_1 + \ln f_2 \right]$$

& $d\vec{p}_1\, d\vec{p}_2 = d\vec{p}_1{}'\, d\vec{p}_2{}'$

$\vec{p}_1$ & $\vec{p}_2$ are the images of $\vec{p}_1{}'$ & $\vec{p}_2{}'$ by $\Rightarrow$ $\vec{p}_1 = (\vec{p}_1{}')'$

$\vec{p}_2 = (\vec{p}_2{}')'$

Changing dummy variables $\vec{p}_1{}', \vec{p}_2{}'$ to $\vec{p}_1, \vec{p}_2$, we get

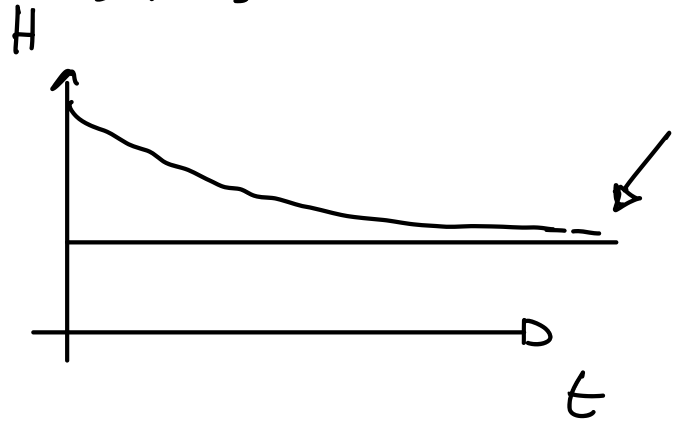
$$② _b = \frac{1}{2} \int d\vec{p}_1\, d\vec{q}\, d\vec{p}_2\; d\sigma(\vec{p}_1, \vec{p}_2 \to \vec{p}_1{}', \vec{p}_2{}')\, |\vec{v}_1 - \vec{v}_2|\, (f_1 f_2 - f_1' f_2')\,[\ln f_1' + \ln f_2']$$

$$② = \frac{1}{2}(②a + ②b)$$

$$\frac{dH}{dt} = \frac{1}{4} \int d\vec{p}_1\, d\vec{q}\, d\vec{p}_2\, d\sigma\, |\vec{v}_1 - \vec{v}_2|\, [f_1' f_2' - f_1 f_2]\,[\ln f_1 f_2 - \ln f_1' f_2']$$

have opposite signs $\Rightarrow ② \leq 0$

$\Rightarrow$ H is a monotonically decreasing function.



Q: what is the steady state?

(i) From the pset: Minimize H under constraints $\Rightarrow$ equilibrium.

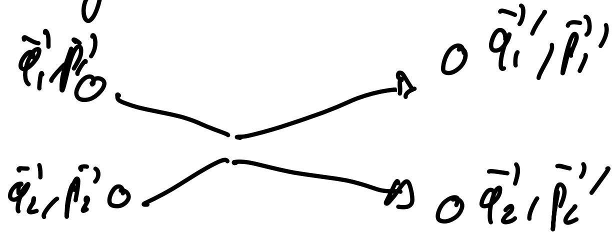
(ii) Is this the only solution?

From H theorem, we need $\forall\, q, p_1, p_2 \quad f_1 f_2 = f_1' f_2'$

$\Leftrightarrow \ln f_1(\vec{p}_1, \vec{q}) + \ln f_1(\vec{p}_2, \vec{q}) = \ln f_1(\vec{p}_1{}', \vec{q}) + \ln f_1(\vec{p}_2{}', \vec{q})$

$\Rightarrow \ln f_1$ is an additive quantity that is conserved during collisions

Five such quantities exist:



* particle number $\quad 1 + 1 \to 1 + 1$
* momentum components $\quad P_{1,\alpha} + P_{2,\alpha} = P'_{1,\alpha} + P'_{2,\alpha}$
* kinetic energy $\quad \vec{P}_1^{\,2} + \vec{P}_2^{\,2} = \vec{P}_1^{\,\prime 2} + \vec{P}_2^{\,\prime 2}$

$$\Rightarrow \ln f_1(\vec{p},\vec{q}) = \gamma(\vec{q}) - \vec{\alpha}(\vec{q}) \cdot \vec{p} - \beta(\vec{q}) \frac{\vec{p}^{\,2}}{2m}$$

Local equilibrium

$$f_1^{LEQ}(\vec{p},\vec{q}) = \tilde{\gamma}(q)\, e^{-\vec{\alpha}(\vec{q})\cdot\vec{p} - \beta(\vec{q})\left[\frac{\vec{p}^{\,2}}{2m} + U(\vec{q})\right]}$$

* H is a decreasing function of time.
* $\forall\ \tilde{\gamma}(q), \vec{\alpha}(q), \beta(\vec{q}), \frac{d}{dt} H\left[f_1^{LEQ}\right] = 0 \Rightarrow$ lots of fixed points of $\underline{\underline{H}}$

Q: Is f^{LEQ} a fixed point of the Boltzmann equation?

$$\partial_t f(\vec{q},\vec{p}_1,t) = \underbrace{-\{f_1, H_1\}}_{\text{relax on } t\sim\tau_F} + \underbrace{C(f,f)}_{}(\vec{q},\vec{p}_1,t)$$

$$= \underbrace{\int d^3\vec{p}_2\, d\sigma\, |\vec{v}_1 - \vec{v}_2| \left(f_1' f_2' - f_1 f_2\right)}_{\text{relax on } t \sim \tau_{MFP}}$$

If $f = f_1^{LEQ}$, then $C(f,f) = 0 \Rightarrow$ invariant by collisions

① $\{f^{LEQ}, H_1\} = 0$ if $\tilde{\gamma}$ & β are constant & $\vec{\alpha} = 0$

Proof: $\{f^{LEQ}, H_1\} = \frac{\partial f^{LEQ}}{\partial \vec{q}} \cdot \frac{\vec{p}}{m} - \frac{\partial f^{LEQ}}{\partial \vec{p}} \cdot \frac{\partial U(\vec{q})}{\partial \vec{q}}$

$$= -\beta \frac{\partial U}{\partial \vec{q}} \cdot \frac{\vec{p}}{m} f_1 - \left[-\beta \frac{\vec{p}}{m} f_1 \cdot \frac{\partial U(\vec{q})}{\partial \vec{q}} \right] = 0$$

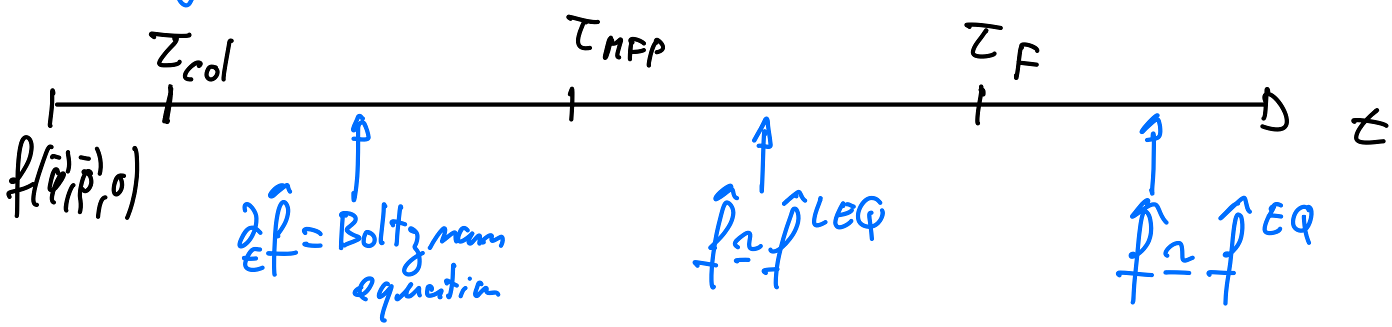
Global equilibrium: $f^{EQ} = \gamma e^{-\beta\left[\frac{\vec{p}^2}{2m} + U(\vec{q})\right]}$

(II) $\{f^{LEQ}, H_1\} \neq 0$ generically if $\tilde{\gamma}(q), \beta(q)$ not constant or $\vec{\alpha} \neq 0$

then $\partial_t f^{LEQ} = - \underbrace{\{f^{LEQ}, H_1\}}_{\neq 0}$

$\Rightarrow$ evolution over $\sim \tau_F$

Summary:



* Collisions equilibrate f locally by making it converge close to f^{LEQ} over t as soon as $t \gg \tau_{MFP}$

* Then, slower evolution to f^{EQ} $\Rightarrow$ Q: HOW??

Comment: The 1-body distribution equilibrates within the canonical ensemble $\Rightarrow$ A priori surprising!

$\Rightarrow$ Because the $N-1$ other particles act as a thermostat.

Q: Why is $V(\vec{q}_i - \vec{q}_j)$ not entering the steady-state??

$$H = \sum_i \Big\{ \underbrace{\frac{\vec{p}_i^{\,2}}{2m} + U(q_i)}_{\sim O(1)} + \underbrace{\sum_{j\neq i} \frac{1}{2} V(\vec{q}_i - \vec{q}_j)}_{\sim n d^3 \ll 1} \Big\} \simeq \sum_i \frac{\vec{p}_i^{\,2}}{2m} + U(q_i)$$

& $P^{EQ}(\{\vec{q}_i, \vec{p}_i\}) \propto e^{-\beta H_1}$ $\Rightarrow$ Ideal gas

$\rightarrow$ enough collisions to equilibrate

$\rightarrow$ too few to alter the statistics

2.4) Transport properties & hydrodynamic equations

2.4.1) Relaxation of the phase space density

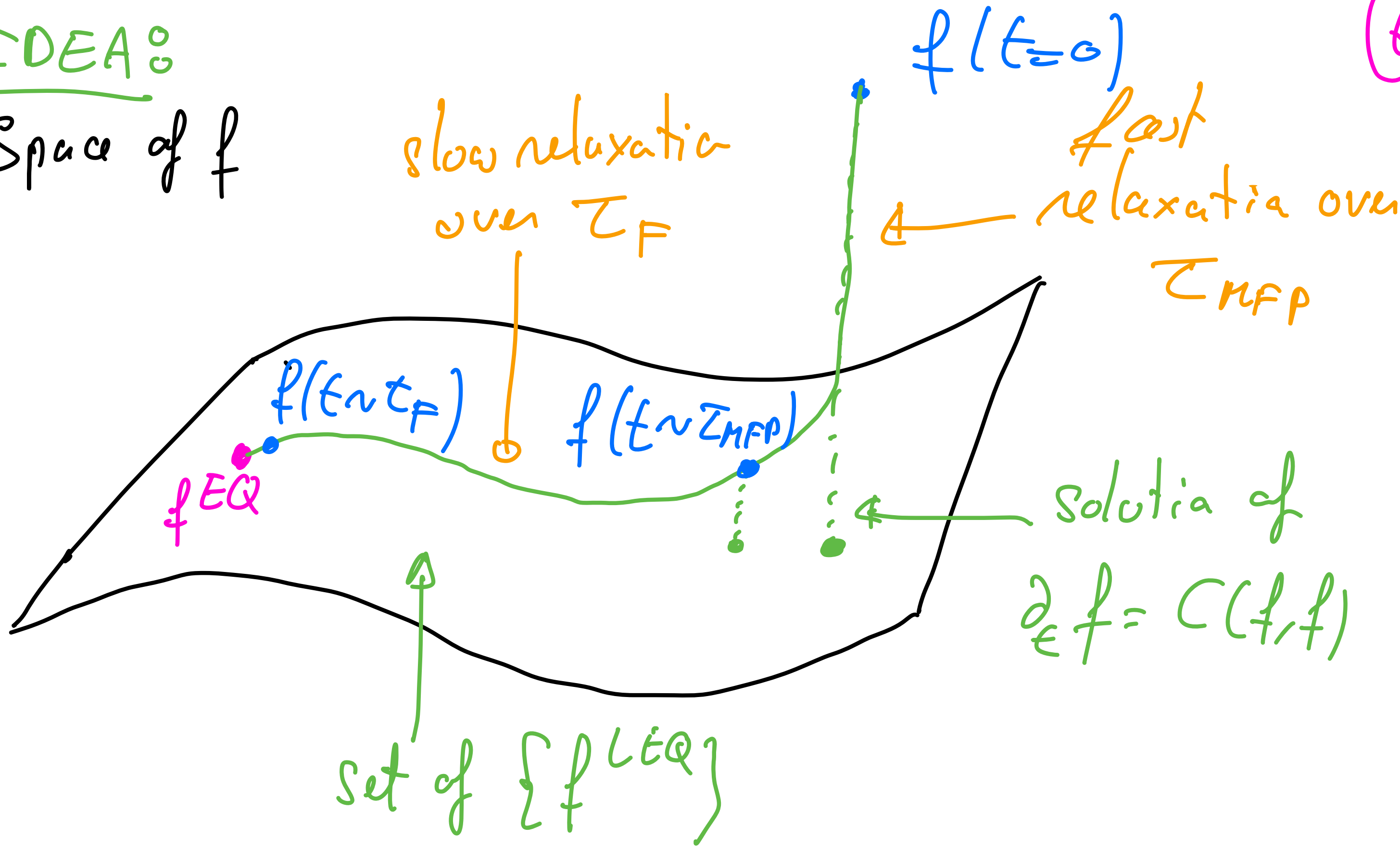
Let us characterize the relaxation of f to f^{EQ}.

We are interested in bulk transport properties

$\Rightarrow U = 0$

IDEA:

Space of f



Q: How to formalise that?

$$\text{BE:}\quad \partial_t f + \underbrace{\frac{\partial f}{\partial \vec{q}} \cdot \frac{\vec{p}}{m}}_{\equiv \frac{1}{\tau_F} L_F f} = \underbrace{C(f,f)}_{\equiv \frac{1}{\tau_{MFP}} \hat{C}(f,f)}$$

where $L_F f = \tau_F \frac{\vec{p}}{m} \cdot \frac{\partial}{\partial \vec{q}} f \underset{\tau_F \to 0}{\sim} \mathcal{O}(1)$

& $\hat{C}(f,f) = \tau_{MFP}\, C(f,f) \underset{\tau_{MFP} \to 0}{\sim} \mathcal{O}(1)$

We have made the scaling with time explicit

We want to measure the relaxation of f over time-scales of order $t \gtrsim \tau_F \Rightarrow t = \hat{t}\,\tau_F$, with $\hat{t} \sim \mathcal{O}(1)$

$$\Rightarrow \frac{\partial}{\partial t} = \frac{1}{\tau_F}\frac{\partial}{\partial \hat{t}}$$

$$(BE)\times\tau_F : \partial_{\hat{t}} f + L_F f = \frac{1}{\varepsilon}\,\mathcal{C}(f,f) \quad (*) \qquad ;\ \varepsilon = \frac{\tau_{MFP}}{\tau_F} \text{ small.}$$

Perturbation theory:

$$f(\vec{q},\vec{p},t) = f_0(\vec{q},\vec{p},t) + \varepsilon f_1(\vec{q},\vec{p},t) + \mathcal{O}(\varepsilon^2)$$

$$(*) \Rightarrow \partial_{\hat{t}} f_0 + \varepsilon\,\partial_{\hat{t}} f_1 + L_F f_0 + \varepsilon L_F f_1 = \frac{1}{\varepsilon} C(f_0,f_0) + C(f_0,f_1) + C(f_1,f_0)$$

Order by order:

$\mathcal{O}(\frac{1}{\varepsilon})$ $\quad 0 = C(f_0,f_0) \Rightarrow f_0 = f^{LEQ}$

$\mathcal{O}(1)$ $\quad \partial_{\hat{t}} f_0 + L_1 f_0 = C(f_0,f_1) + C(f_1,f_0) \Rightarrow$ dynamics of f_0

$\mathcal{O}(\varepsilon)$ $\quad \partial_{\hat{t}} f_1 + \dots \Rightarrow$ dynamics of f_1

Leading order approximation

$$f(\vec{q},\vec{p},t) = f^{LEQ}(\vec{p},\vec{q},t) = \gamma(q)\,e^{-\vec{\alpha}(\vec{q})\cdot\vec{p} - \beta(\vec{q})\left[\frac{\vec{p}^{\,2}}{2m} + U(\vec{q})\right]}$$

Idea: the gas equilibrates locally over $\tau \sim \tau_{MFP}$